

THE ASYMPTOTIC VARIETY OF A PINCHUK MAP AS A POLYNOMIAL CURVE

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ABSTRACT. The asymptotic variety of a counterexample of Pinchuk type to the strong real Jacobian conjecture is explicitly described by low degree polynomials.

1. INTRODUCTION

Let the polynomial map $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the Pinchuk map of total degree 25 considered in [Cam96, Ess00, Cam98, Cam08]. Its asymptotic variety, $A(F)$, a closed curve in the image (P, Q) -plane, was computed in [Cam96]. It is depicted below using differently scaled axes. It intersects the vertical axis at $(0, 0)$ and $(0, 208)$ and its leftmost point is $(-1, -163/4)$.

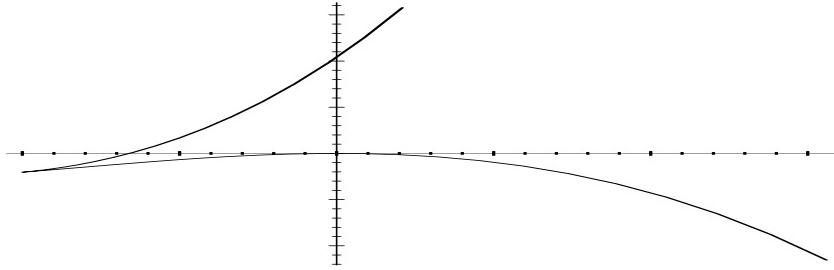


FIGURE 1. The asymptotic variety of the Pinchuk map F .

This brief note will show that $A(F)$ has the bijective polynomial parametrization by $s \in \mathbb{R}$:

$$(1) \quad P(s) = s^2 - 1$$

$$(2) \quad Q(s) = -75s^5 + \frac{345}{4}s^4 - 29s^3 + \frac{117}{2}s^2 - \frac{163}{4}$$

and that its points satisfy the minimal equation

$$(3) \quad (Q - (345/4)P^2 - 231P - 104)^2 = (P + 1)^3(75P + 104)^2$$

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In particular, only one point on the curve satisfies $P = -1$, and that point is its only singular point. Also, there is a single point with $P = -104/75$ which satisfies Eqn (3). That point is not on the curve itself, because Eqn (1) implies $P \geq -1$, but belongs to its Zariski closure.

Exactly parallel facts were established in [Gwo00] for a different Pinchuk map \tilde{F} of total degree 40. In fact, any two Pinchuk maps have essentially the same behavior and asymptotic variety, differing only by a triangular polynomial automorphism of the image plane.

A parametrization of $A(F)$ appeared in the unpublished preprint [Cam98]. The derivation is shortened here. F is a useful reference example, because of the simplicity and low degree of the explicit equations for $A(F)$.

2. PINCHUK MAPS

Pinchuk maps are certain polynomial maps $F = (P, Q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that have an everywhere positive Jacobian determinant $j(P, Q)$, and are not injective [Pin94]. The polynomial $P(x, y)$ is constructed by defining $t = xy - 1, h = t(xt + 1), f = (xt + 1)^2(t^2 + y), P = f + h$. Note that $\deg h = 5, \deg f = 10$, so $\deg P = 10$. The polynomial Q varies for different Pinchuk maps, but always has the form $Q = -t^2 - 6th(h + 1) - u(f, h)$, where u is an auxiliary polynomial in f and h , chosen so that $j(P, Q) = t^2 + (t + f(13 + 15h))^2 + f^2$. As in [Cam96, Ess00], choose specifically

$$(4) \quad u = 170fh + 91h^2 + 195fh^2 + 69h^3 + 75fh^3 + \frac{75}{4}h^4.$$

Then $\deg F = \deg Q = 25$.

Suppose $\tilde{F} = (P, \tilde{Q})$ is a different Pinchuk map defined using \tilde{u} . Observe that $A = Q - \tilde{Q} = u(f, h) - \tilde{u}(f, h)$ lies in $\mathbb{R}[P, h]$ and satisfies $j(P, A) = 0$, since $j(P, \tilde{Q}) = j(P, Q)$. By [Zak71] or [Ess00, Thm. 1.2.25], the subalgebra $\mathbb{R}[P, A] \subset \mathbb{R}[x, y]$ is generated by a single element. In $\mathbb{R}[P, h]$ that generator must be a degree 1 polynomial in P alone, since P and h are algebraically independent. So $A \in R[P]$ and $\tilde{F} = T \circ F$ for a triangular polynomial automorphism $T(x, y) = (x, y + S(x))$.

The “original” Pinchuk map of [Gwo00] is defined by adding, not subtracting, $(1/4)f(75f^3 + 300f^2h + 450fh^2 + 276f^2 + 828fh + 48h^2 + 364f + 48h)$, which is thus $-\tilde{u}$. Clearly $\deg \tilde{F} = \deg \tilde{Q} = 40$.

Note that no Pinchuk map can have degree less than 25. For P has degree 10, and so, if $\tilde{Q} = Q + S(P)$, there is no way to cancel the terms of degree 25 without introducing terms of yet higher degree.

3. ASYMPTOTIC BEHAVIOR

The points $(-1, -163/4)$ and $(0, 0)$ of $A(F)$ have no inverse image under F , all other points of $A(F)$ have one inverse image, and all points of the image plane not on $A(F)$ have two. See [Cam96, Cam08].

If the two omitted points are deleted from $A(F)$, we are left with three curves. Since the curves tend to infinity or to an omitted point at either end, their inverse images tend to infinity at both ends. So they partition their complement, $\mathbb{R}^2 \setminus F^{-1}(A(F))$, into four simply connected domains. These domains are mapped homeomorphically to their images, two each to the domains on either side of $A(F)$. See [Cam98].

Suppose $\tilde{F} = (P, \tilde{Q}) = T \circ F$ is a different Pinchuk map, with asymptotic variety $A(\tilde{F})$. From the definition of the asymptotic variety of a polynomial map as the set of finite limits of the map along curves that tend to infinity ([Per96, Per98]), or equivalently, the points at which the map is not proper ([Jel93, Jel02]), it follows that $A(\tilde{F}) = T(A(F))$. The behavior is that of F , up to the triangular automorphism T of the image plane.

Also, note that the partition of the (x, y) -plane into three curves and four domains is exactly the same as for F . See [Gwo00] for a graphic depiction.

4. EQUATIONS FOR $A(F)$

Also from previously cited work, a general level set $P = c$ in the (x, y) -plane has a rational parametrization

$$\begin{aligned} x(h) &= \frac{(c-h)(h+1)}{(c-2h-h^2)^2} \\ y(h) &= \frac{(c-2h-h^2)^2(c-h-h^2)}{(c-h)^2}, \end{aligned}$$

which can be obtained by solving $P = c$ for x and then y , and can be readily verified by substitution into the defining equations $t = xy - 1, h = t(xt + 1), f = (xt + 1)^2(t^2 + y), P = f + h$. Note that $h(x(h), y(h))$ does indeed simplify to just h , and $P(x(h), y(h))$ to just c .

Temporarily ignore the special cases $c = -1$ and $c = 0$, for which different parametrizations apply. At poles $c = h$, it is easy to check that $Q = -t^2 - 6th(h+1) - u(f, h)$ is infinite. Indeed, $f = c - h = 0$ and the dominant pole is $-h^4(h+1)^2/(c-h)^2$ with a nonzero numerator by the restriction on c . In contrast, as h tends to a pole $c = h^2 + 2h$ we find that Q approaches a finite limit. The limit condition on c and h can also be stated as $h = -1 \pm \sqrt{1+c}$. This yields two points of $A(F)$ on a vertical line $P = c$ when $c > -1$ and $c \neq 0$, and no points when $c < -1$.

In more detail, $x(h)y(h) = (h+1)(c-h-h^2)(c-h)^{-1}$. If we substitute $c = h^2 + 2h$, the expression simplifies to $h(h+1)h^{-1}(h+1)^{-1}$. Here too, $h \neq 0$ and $h \neq -1$ by the restriction on c , so the ratio is defined and equal to 1. This only means that as h tends to such a pole, xy tends to 1. In the limit $t = 0, f = c - h = h^2 + h$ and thus $Q = -t^2 - 6th(h+1) - u(f, h) = -u(h^2 + h, h)$. Since $A(F)$ is closed in the Euclidean topology, it contains the three sofar missing points with $P = -1$ or $P = 0$ and the same equations hold there by continuity. That is all of $A(F)$, because every point not obtained is, by section 3, known not to be an asymptotic value of F .

The polynomial parametrization $(P(h), Q(h)) = (h^2 + 2h, -u(h^2 + h, h))$ is clearly a bijection from \mathbb{R} onto $A(F)$. This works (with the appropriate u) for any Pinchuk map; it is the form reported in [Gwo00].

Using $s = h + 1$ as a parameter instead and the specific auxiliary polynomial $u(f, h)$ in Eqn (4) yields Eqn (1) and Eqn (2). The choice of s simplifies the calculation of the gradient of the parametrization and of points on $A(F)$. From Eqn (2)

$$Q - \frac{345}{4}s^4 - \frac{117}{2}s^2 + \frac{163}{4} = -s(75s^4 + 29s^2).$$

Squaring both sides and substituting $P + 1$ for s^2 yields Eqn (3). Expand and rewrite Eqn (3) as an implicit polynomial equation $B(P, Q) = 0$, quadratic in Q . B cannot have a factor that is a nonconstant polynomial in P , because the coefficient of Q^2 in B is 1. Nor can B have two factors linear in Q . At least one such factor, say $Q - K(P)$ for a polynomial K , would have to be identically 0 on $A(F)$, yet some vertical lines $P = c$ do not intersect $A(F)$. So B is irreducible and therefore its set of zeroes is the Zariski closure of $A(F)$.

5. DOUBLE ASYMPTOTIC IDENTITIES

Ronen Peretz championed a simpler way of finding parametrization equations such as Eqn (1) and Eqn (2) [Per96, Per98]. A double asymptotic identity for F is an equation $F(R(x, y)) = G(x, y)$ for a rational (but not polynomial) map R and a polynomial map G .

Consider $R = (x^{-2}, yx^3 + x^2)$. As $t \circ R = xy, h \circ R = (x + y)y, f \circ R = (x + y)^2(y^2 + xy + 1)$, the map $G = (P \circ R, Q \circ R)$ is polynomial.

As x tends to zero for a fixed y , the point $R(x, y)$ tends to infinity, describing a curve along which F tends to the finite limit point $G(0, y)$. So $G(0, y) = (y^4 + 2y^2, -u(y^4 + y^2, y^2))$ is a parametrization of (some of the points of) $A(F)$.

Comparing with the bijective parametrization by h of the previous section, it is evident that this parametrization covers only the points $h \geq 0$. Each such point is obtained twice, except for $(P, Q) = (0, 0)$, where the parametrization reverses course. A similar parametrization covering the points $h \leq 0$ of $A(F)$ arises from the alternate choice of $(-x^{-2}, yx^3 - x^2)$ for R [Cam98].

The computations of the previous section can be recast into a rational identity of the form $F(R(x, y)) = G(x, y)$ that provides the bijective parametrization. However, G is not a polynomial map, but rather a rational map with $G(0, y)$ defined for all but finitely many values of y and polynomial in y . The two exceptional values of y correspond to the special cases $P = -1$ and $P = 0$.

6. RELATION TO THE JACOBIAN CONJECTURE

A weak Jacobian conjecture for polynomial maps of \mathbb{R}^2 to itself is that a Keller map (nonzero constant Jacobian determinant) is injective. This is weaker than the standard Jacobian conjecture $\text{JC}(2, \mathbb{R})$, even though injectivity implies bijectivity here, because the inverse is not required to be a polynomial map.

Remark. Over \mathbb{C} this distinction does not exist, since any inverse map is birational and everywhere defined, hence polynomial. Both conjectures for \mathbb{R}^2 would follow from $\text{JC}(2, \mathbb{C})$. Note that $\text{JC}(2, \mathbb{R})$ is not known to imply $\text{JC}(2, \mathbb{C})$.

A general feature of the Pinchuk map F that conflicts with the Keller condition is that radial similarity of Newton polygons fails. $N(P) = N(x^6y^4 + x^2 + y)$, a quadrilateral, while $N(Q) = N(x^{15}y^{10} + x^3y^4 + x^5 + y)$, a five-sided polygon. For \tilde{F} , though, $N(\tilde{Q}) = N(x^{24}y^{16} + x^8 + y^4)$, a 4-fold radial expansion of $N(P)$. All these polygons have no edge of negative slope.

For a nonsingular map $f = (p, q) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, whether polynomial or not, $j(p, q)$ is the rate of change of q along the level curves of p , parametrized as the flow of the Hamiltonian vector field $H(p) = (-\partial p / \partial y, \partial p / \partial x)$. If f is polynomial the local flow $(x(t), y(t))$ with initial condition $x(0) = a, y(0) = b$ is at least real analytic. If f is also a Keller map, it has a polynomial inverse if the power series expansions for x and y have infinite radius of convergence for even a single point $(a, b) \in \mathbb{R}^2$,

in which case the flow is actually polynomial for any $(a, b) \in \mathbb{R}^2$. That follows from the corresponding result for complex Keller maps [Cam97, Thm 3.2], by treating f as a complex polynomial map and (a, b) as a point of \mathbb{C}^2 .

7. ACKNOWLEDGMENTS

Eqn (4) was first circulated by Arno van den Essen in an email message to a number of colleagues in June 1994. I made two blunders and a significant typographical error in describing the associated Pinchuk map F in [Cam96]. The errors were corrected in [Cam98] and belatedly for the official record in [Cam08]. Janusz Gwoździewicz significantly clarified the relationship between different Pinchuk maps for me.

REFERENCES

- [Cam96] L. A. Campbell. Partial properness and real planar maps. *Appl. Math. Lett.*, 9(5):99–105, 1996.
- [Cam97] L. Andrew Campbell. Jacobian pairs and Hamiltonian flows. *J. Pure Appl. Algebra*, 115(1):15–26, 1997.
- [Cam98] _____. Picturing Pinchuk’s plane polynomial pair. arXiv:math.AG/9812032, December 1998.
- [Cam08] _____. Erratum to the Pinchuk map description in: “Partial properness and real planar maps” [Appl. Math. Lett. 9 (1996), no. 5, 99–105. Appl. Math. Lett., 21(5):534–535, 2008.]
- [Ess00] Arno van den Essen. *Polynomial automorphisms and the Jacobian conjecture*, volume 190 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2000.
- [Gwo00] Janusz Gwoździewicz. A geometry of Pinchuk’s map. *Bull. Polish Acad. Sci. Math.*, 48(1):69–75, 2000.
- [Jel93] Zbigniew Jelonek. The set of points at which a polynomial map is not proper. *Ann. Polon. Math.*, 58(3):259–266, 1993.
- [Jel02] _____. Geometry of real polynomial mappings. *Math. Z.*, 239(2):321–333, 2002.
- [Per96] Ronen Peretz. The variety of asymptotic values of a real polynomial étale map. *Journal of Pure and Applied Algebra*, 106:102–112, 1996.
- [Per98] _____. The geometry of the asymptotics of polynomial maps. *Israel J. Math.*, 105:1–59, 1998.
- [Pin94] Sergey Pinchuk. A counterexample to the strong real Jacobian conjecture. *Math. Z.*, 217(1):1–4, 1994.
- [Zak71] Abraham Zaks. Dedekind subrings of $k[x_1, \dots, x_n]$ are rings of polynomials. *Israel J. Math.*, 9:285–289, 1971.

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